HISTORIA MATHEMATICA 2 (1975), 601-602

HOW MATHEMATICS PROGRESSES

BY JOAQUIN B. DIAZ
RENSSELAER POLYTECHNIC INSTITUTE

[This is a summary of the actual talk, prepared by the editor and approved by Professor Diaz.]

Professor Diaz gave three illustrations of the influence of the "family" on the historical development of mathematics, having in mind the usual German description of a Ph.D. advisor as the "Doctor Father" of the Ph.D. candidate.

The first example concerned Young's inequality

$$ab \leq \int_0^a f(x)dx + \int_0^b f^{-1}(y)dy, \tag{1}$$

where f^{-1} is the inverse of f, a strictly increasing function. The usual proof is by an intuitive appeal to the areas [E2]. Curiously, although such "geometric" proofs have not been accepted in analysis since the time of Weierstrass, (1) was given its first elementary rigorous analytic proof only recently by Diaz and his student Metcalf [E10].

The second example concerned new proofs of other old inequalities, also by Diaz and his "academic son" Metcalf. In [E9], they showed that four inequalities occurring in scattered literature can be deduced from a single inequality, which includes all four special cases and which is much easier to prove than any of the four inequalities mentioned. This illustrates the gradual process of unification and simplification which, though usually little publicized, is so necessary for the continuing progress of mathematics.

The third example concerned Peano's classical local existence theorem for solutions of the "simplest" ordinary differential equation $y' = f(x,y)$, $y(x_0) = x_0$, where the real-valued function f of the two real variables, x and y, is supposed to be continuous on a rectangle centered at the point x_0, y_0. It is well known that one of the basic existence proofs of a solution of this Cauchy problem is due to Peano (See [E23]; [E24]; and G. Mie, *Math. Annalen* 43 (1893), pp. 553-568) and is based on a "numerical" method called the Euler-Cauchy polygon method, the essential idea of which goes back to Euler in the 18th century. This differential equation may also be considered in the case when x remains a real variable, y is a vector in a Banach space, and the function f has values in the same Banach space. In this case the question has already been raised in the literature of whether the Peano theorem holds (that is to say, whether there exists a solution, not necessarily unique, when the vector-valued function f is merely continuous). As a matter of fact, one of the participants of this conference, Professor Dieudonné, has given an example showing that the Peano theorem does not hold for an arbitrary Banach space.

A solution to this problem has been given recently in [E8]. Here Diaz and Bownds [The authorship is of interest because Bownds was a student of Metcalf, and consequently an "academic grandson" of Professor Diaz. Historians of mathematics should try to bring out more systematically the role played by such "family trees." -- Ed.] showed that the Euler-Cauchy polygon method can be used to prove the existence of a solution, in the vector-valued case. Of course, we have to make an additional restriction on f, in order to take into account Dieudonné's example, and the restriction we make is that the range of the function f be compact. This proof is of interest because it does not assume any previous knowledge of a theory of integration of vector-valued functions of a real variable, and of particular interest because, if the function f does not depend upon y, then the proposed Cauchy initial value problem reduces precisely to the problem of integrating the vector-valued function $f(x)$.

The final comment by Diaz was strictly philosophical. It was that "Mathematics is an experimental science". Diaz first heard this dictum a year ago from a philosopher, Professor Sidney Axinn of Temple University. Since then, he has come to agree with it, albeit somewhat reluctantly! He contrasted proofs in Euclid's *Geometry*, which use axioms agreed to in advance, with the process followed in a physics course, where one proves that atmospheric pressure is about 76 mm. of mercury by an experiment like Torricelli's. He asked his auditors to think about the similarities and the differences between these two kinds of "proof".